Journal of Engineering Mathematics, Vol. 8, No. 4, October 1974
Noordhoff International Publishing – Leyden
Printed in The Netherlands

Irrotational axisymmetric flow about a prolate spheroid in cylindrical duct

T. MILOH

Department of Fluid Mechanics and Heat Transfer, School of Engineering, Tel-Aviv University, Israel

(Received April 16, 1974)

SUMMARY

A solution to the problem of potential flow about a prolate spheroid placed axially symmetric in a circular duct has been derived. The solution is in the form of a distribution of vortex rings over the surface of the spheroid. The vortex strength is expressed in terms of an infinite series of Legendre polynomials and the analysis yields an infinite set of equations for determining the coefficients of this series. An expression for the velocity distribution on the surface of the spheroid as well as the longitudinal added mass coefficients of the spheroid are derived in terms of the coefficients of the Neumann series expansion of the vortex sheet strength.

Numerical results are presented for various spheroids and different blockages. Also given is a comparison between the present method and few available approximate methods.

1. Introduction

The first step in determining the flow field about an axisymmetric body in cylindrical duct is the solution of the incompressible potential flow problem. The presence of the wall will alter the velocity and pressure distributions about the body as compared with the case where the flow field is externally unbounded. The so-called "blockage effect" has to be considered when potential flow and boundary layer calculations are performed for axisymmetrical models in water or wind tunnels. The same problem arises also in the design of nozzles and diffusers with an axisymmetrical core, in the design of turbomachinery elements, and in tube transportation related problems.

Several methods are available for the calculation of the potential flow about bodies of revolution placed axisymmetrically in a cylindrical duct. Levine's method [1] employs a series of Bessel functions which satisfy the Neumann boundary condition on the circular wall. The series given by Levine is essentially an expression for Green's function of a unit source placed on the axis of the tube. The Green's function may also be expressed in terms of an integral of Bessel functions as shown by Watson [2]. This method has been employed by Satija [3] for the computation of the velocity distribution about a body of revolution and by Goodman [4] in analysing the stability of a slender body in a circular tube. It may be mentioned that Satija's work yields only a first-order blockage correction. A procedure for determining a higher-order blockage correction has been proposed by Landweber and Gopalakrishnan [5] by solving a pair of integral equations; one for an axial doublet distribution and the second for the velocity distribution on the body surface. Mathew [6], for the velocity distribution on a body of revolution, used a vortex sheet to formulate integral equations of both the first and second kind although he employed only that of the second kind. Klein and Mathew [7] chose to represent both the duct and the body by a continuous distribution of ring vortices. The vortex strengths were then given by the solution of two coupled Fredholm integral equations of the second kind. Still another method for solving the potential flow for axisymmetric body-duct configuration, was that suggested by Kux and Wieghardt [8] which employed the Hess and Smith [9] method and a distribution of sources over the circular wall.

The final and major step, in all of the above mentioned methods, was the numerical solution of an integral equation for the axial source distribution, axial doublet distribution, vortex sheet or source ring. The potential flow about a prolate spheroid has been used to demonstrate the applicability of each one of these methods [3, 4, 5, 8].

The numerical solution of the integral equation has to be performed with special care due to the sharp peaking of the kernel [5]. Any attempt to ignore this effect and use a quadrature formula of moderate order, may result in appreciable inaccuracy, as is readily detected from the plot of the numerical solution in [6]. Moreover, those methods which employ an integral equation of the first kind [3, 5, 6] may very well diverge for small blockages since it is known that a solution of an integral equation of the first kind does not exist in general [10].

In order to examine which method is superior as far as accuracy, applicability for small blockages and computer time are concerned, it is necessary to compare it against exact solution (if available). It is well known that the potential problem of incompressible unbounded flow past a spheroid bears an exact solution [11]. Is it possible to obtain an "exact" solution also for the case where the prolate spheroid is placed axisymmetrically in a circular tube? In addition to the eccentricity of the spheroid the solution will depend on the channel diameter. "Exact" should be interpreted as an exact solution in the least square sense. This is because an exact solution of an integral equation of the first kind does not exist in general and every solution which is obtained should be considered as an approximation in the Lebesque sense.

Unfortunately an exact solution for the potential flow about a spheroid in a duct is not available. Separation of variables, which is the usual procedure for solving a potential boundary value problem for a spheroidal boundary, fails because the circular wall cannot be expressed in terms of one of the orthogonal spheroidal coordinates. Also the method of spheroidal harmonics which assumes *a priori* that the external potential may be expressed as a linear combination of an infinite set of exterior and interior spheroidal harmonics, is not applicable for the present case because of the condition at infinity. An attempt to solve this problem will be made here by expressing the solution of the integral equation in terms of a Neumann series whose coefficients are obtained by solving a set of linear equations.

Since in the present case our interest is in the velocity distribution (or pressure distribution) about the spheroid. [3, 5, 6, 7, 8], it is more convenient to use Landweber's [12] formulation in terms of an integral equation for a vortex sheet, modified by Mathew [6] for a flow in a tube. The vortex sheet is expressed in terms of a Neumann series of Legendre polynomials and the Fredholm integral equation of the first kind is solved for the coefficients of this series. By employing the orthogonality properties of the Legendre polynomials it is possible to obtain an infinite set of linear equations for the coefficients of the series for the velocity distribution about the spheroid. These coefficients depend on the diameter of the duct and on the eccentricity of the spheroid. Asymptotic solutions for large blockages and expressions for the longitudinal added mass coefficients have been derived. Numerical comparison between the present method and the approximate methods given in [3, 5, 8] is also presented.

2. Formulation of the problem

Consider the incompressible and irrotational flow of an inviscid fluid about a prolate spheroid, placed axially in a circular tube. In a cylindrical coordinate system x and r, with an origin at the centroid of the spheroid, let x measure distance along the duct centerline in the direction of the ambient velocity, here denoted by U. In the same coordinate system, the equation of the impermeable cylindrical wall is given by $r=a$, where a denotes the radius of the tube. For the other solid boundary, i.e. the spheroid, it is convenient to use axisymmetric prolate spheroidal coordinates μ and ζ such that

$$x=\mu\zeta\,,\qquad r=(\zeta^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\,, \tag{1}$$

where it is assumed that the two foci of the spheroid are located at $x=\pm 1$. In the spheroidal coordinate system the equation of the spheroidal surface is given by $\zeta=\zeta_0$, where ζ_0 is the inverse of the eccentricity of the spheroid. Since our interest, in the present paper, is to solve for the velocity (or the pressure distribution about the spheroid), it is convenient to formulate the problem in terms of vortex sheet distribution over the surface of the spheroid. The jump property of a vortex sheet implies that the velocity at the exterior part of the surface is equal to the vortex sheet strength when the velocity at the interior part of the surface is chosen to be zero.

In order to perform the integration along the x-axis, rather than on the body surface it is convenient to introduce a new variable, $\gamma(x)$ related to the velocity on the surface $u(x)$ by the following definition:

$$u(x) = \gamma(x)\cos\theta\,, \tag{2}$$

where θ is the angle between the tangent to the body and the x-axis. For the spheroidal surface given by $\zeta=\zeta_0=$const., equation (2) reads

$$u(x) = \zeta_0\gamma(x)\left(\frac{\zeta_0^2-x^2}{\zeta_0^4-x^2}\right)^{\frac{1}{2}}. \tag{3}$$

An integral equation for the vortex sheet strength has been derived by Mathew [6]. Here we will give a different proof which appears to be much shorter and simpler than Mathew's derivation.

The basic potential function for a unit sink placed at the origin on the axis of the circular duct which satisfies a Neumann boundary condition on the duct wall $r=a$ is [3]

$$\phi_s(x, r) = (x^2+r^2)^{-\frac{1}{2}} + \frac{2}{\pi}\int_0^\infty K_1(ta)\frac{I_0(tr)}{I_1(tr)}\cos(tx)\,dt\,, \tag{4}$$

where I_n and K_n denote the modified Bessel functions of nth order of the first and second kind respectively [13].

According to the Biot–Savart law the velocity induced at points on the x-axis by an axisymmetric vortex ring of unit strength and radius r centered at the origin is

$$\frac{1}{2}\frac{r^2}{(x^2+r^2)^{\frac{3}{2}}} = -\frac{r}{2}\frac{\partial}{\partial r}(x^2+r^2)^{-\frac{1}{2}}\,. \tag{5}$$

Equation (4) then suggests that the velocity induced by the same vortex ring at points along the x-axis, when it is placed in a circular duct of radius a, is

$$u_s(x, r) = \frac{1}{2}\frac{r^2}{(x^2+r^2)^{\frac{3}{2}}} - \frac{2r}{\pi}\int_0^\infty tK_1(ta)\frac{I_1(tr)}{I_1(ta)}\cos(tx)\,dx\,. \tag{6}$$

Consider now a continuous distribution of vortex rings with strength $\gamma(x)$ over the spheroidal surface. Employing Landweber's [12] method for the condition of zero velocity within the body yields the following integral equation:

$$2U = \int_{-\zeta_0}^{\zeta_0}\frac{\gamma(\xi)r^2(\xi)d\xi}{[(x-\xi)^2+r^2(\xi)]^{\frac{3}{2}}} - \frac{2}{\pi}\int_{-\zeta_0}^{\zeta_0}\int_0^\infty tr(\xi)\gamma(\xi)K_1(ta)\frac{I_1(tr)}{I_1(ta)}\cos\{t(x-\xi)\}\,dt\,d\xi\,. \tag{7}$$

It may be mentioned that (7) is valid for bodies of revolution of arbitrary shape provided that the equation $r(x)$ is prescribed. For the present case,

$$r^2(x) = (\zeta_0^2-1)(1-x^2/\zeta_0^2)\,, \qquad |x| \leqq \zeta_0 \tag{8}$$

and we are concerned with the solution of the Fredholm integral equation of the first kind (7) for the unknown $\gamma(x)$.

The method of solution used is to assume that $\gamma(x)$ may be expressed in terms of a convergent Neumann series of Legendre polynomials [14]

$$\gamma(\xi) = \sum_{n=0}^{\infty} A_n P_n(\mu)\,, \qquad \xi = \zeta_0\mu\,, \tag{9}$$

where the coefficients A_n are functions of the eccentricity of the spheroid and the radius of the duct. In order to solve (9) for the unknowns, A_n, let us denote the two integrals on the right-hand side of (7) by $T_1(x, \zeta_0)$ and $T_2(x, \zeta_0, a)$ respectively.

3. Treatment of single integral

The kernel of the integral $T_1(x, \zeta_0)$ may be written as

$$\frac{r^2(\xi)}{R^3} = \frac{1}{R} + (x-\xi)\frac{\partial}{\partial x}\left(\frac{1}{R}\right), \tag{10}$$

where

$$R^2 = R^2(x, \xi) = (x-\xi)^2 + r^2(\xi), \tag{11}$$

and r is defined by (8).

In addition one has the following expansion [15] for the inverse of the distance between two points expressed as an infinite sum of spheroidal harmonics

$$\frac{1}{R} = \sum_{n=0}^{\infty} (2n+1) P_n(x) P_n(\mu) Q_n(\zeta_0), \tag{12}$$

where Q_n denotes the nth order Legendre function of the second kind. Combining (10) and (12) then yields

$$\frac{r^2(\xi)}{R^3} = \sum_{n=0}^{\infty} (2n+1) P_n(\mu) Q_n(\zeta_0)[P_n(x)+(x-\mu\zeta_0)\dot{P}_n(x)], \tag{13}$$

where the dot denotes differentiation with respect to the argument.

Substitution of (9) and (13) into (7) yields

$$T_1(x, \zeta_0) = \int_{-\zeta_0}^{\zeta_0} \frac{\gamma(\xi) r^2(\xi)}{R^3} d\xi =$$

$$= \zeta_0 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (2n+1) A_m Q_n(\zeta_0) \int_{-1}^{1} P_n(\mu) P_m(\mu)[P_n(x)+(x-\mu\zeta_0)\dot{P}_n(x)] d\mu. \tag{14}$$

Making use of the orthogonality properties of the Legendre polynomials, i.e.

$$\int_{-1}^{1} P_m(\mu) P_n(\mu) d\mu = \begin{cases} 2/(2n+1), & m=n \\ 0, & m \neq n \end{cases} \tag{15}$$

and the following relation, [14],

$$\mu P_n(\mu) = \frac{n}{2n+1} P_{n-1}(\mu) + \frac{n+1}{2n+1} P_{n+1}(\mu), \tag{16}$$

equation (14) may be reduced to

$$T_1(x, \zeta_0) = 2\zeta_0 \sum_{n=0}^{\infty} Q_n(\zeta_0)\Big\{[P_n(x)+x\dot{P}_n(x)] A_n - \frac{n}{2n-1}\zeta_0 \dot{P}_n(x) A_{n-1} - \frac{n+1}{2n+3}\zeta_0 \dot{P}_n(x) A_{n+1}\Big\}. \tag{17}$$

Define, for later use, a coefficient $T_1^m(\zeta_0)$ by

$$T_1^m(\zeta_0) = \int_{-1}^{1} T_1(x, \zeta_0) P_m(x) dx. \tag{18}$$

Introducing (17) in (18) yields

$$T_1^m(\zeta_0) = 4\frac{m+1}{2m+1}\zeta_0 Q_m(\zeta_0) A_m + 2\zeta_0 \sum_{n=0}^{\infty}\Big\{Q_{n+1}(\zeta_0) A_{n+1} - \zeta_0 Q_n(\zeta_0)\left[\frac{n}{2n-1} A_{n-1} + \frac{n+1}{2n+3} A_{n+1}\right]\Big\} \int_{-1}^{1} \dot{P}_n(x) P_m(x) dx. \tag{19}$$

since, [14],

$$x\dot{P}_n(x) = \dot{P}_{n-1}(x) + nP_n(x)\,, \qquad n>0\,. \tag{20}$$

The following expansion is used for $\dot{P}_n(x)$, [14],

$$\dot{P}_n(x) = (2n-1)\,P_{n-1}(x) + (2n-5)\,P_{n-3}(x) + (2n-9)\,P_{n-5}(x) + \ldots \tag{21}$$

hence, from (15) and (21) there results,

$$\int_{-1}^{1} \dot{P}_n(x)\,P_m(x)\,dx = \begin{cases} 2\,, & n = m+2k+1 \\ 0\,, & n \neq m+2k+1 \end{cases} \tag{22}$$

where k is an arbitrary positive integer.

By substituting (15) and (22) into (19), we obtain

$$T_1^m(\zeta_0) = 4\,\frac{m+1}{2m+1}\,\zeta_0 A_m[Q_m(\zeta_0) - \zeta_0 Q_{m+1}(\zeta_0)] + 4\zeta_0 \sum_{k=1}^{\infty} A_{m+2k}\left[Q_{m+2k}(\zeta_0) - \frac{m+2k+1}{2m+4k+1}\,\zeta_0 Q_{m+2k+1}(\zeta_0) - \frac{m+2k}{2m+4k+1}\,\zeta_0 Q_{m+2k-1}(\zeta_0)\right]. \tag{23}$$

Finally, using recurrence formulae for the Legendre polynomials of the second kind [14], equation (23) may be simplified to read

$$T_1^m(\zeta_0) = -\frac{4}{2m+1}\,\zeta_0(\zeta_0^2-1)\,\dot{Q}_{m+1}(\zeta_0)\,A_m - 4\zeta_0(\zeta_0^2-1)\sum_{k=1}^{\infty} Q_{m+2k}(\zeta_0)\,A_{m+2k}\,. \tag{24}$$

4. Treatment of double integral

The double integral in (7), here denoted by $T_2(x, \zeta_0, a)$ may also be expressed as

$$T_2(x,\zeta_0,a) = -\frac{2}{\pi}\,\zeta_0(\zeta_0^2-1)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n \int_0^{\infty}\!\!\int_{-1}^{1} t(1-\mu^2)^{\frac{1}{2}} P_n(\mu)\,\frac{K_1(ta)}{I_1(ta)}\, I_1\left[t(\zeta_0^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\right] \cos\{t(x-\mu\zeta_0)\}\,dt\,d\mu\,, \tag{25}$$

where $\gamma(\xi)$ is replaced by its series expansion (9).

Following Havelock [16], there exists the following relation:

$$\int_{-1}^{1} P_n^m(\mu)\,I_m[t(\zeta_0^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}]\,e^{it\zeta_0\mu}\,d\mu = 2(i)^{n-m} P_n^m(\zeta_0)\,j_n(t)\,, \tag{26}$$

where P_n^m denotes the associated Legendre function defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\,\frac{d^m P_n(\mu)}{d\mu^m}\,; \quad P_n^m(\zeta_0) = (\zeta_0^2-1)^{m/2}\,\frac{d^m P_n(\zeta_0)}{d\zeta_0^m}\,. \tag{27}$$

Here j_n denotes the nth order spherical Bessel function [13], and i denotes the square root of minus one.

Application of the recurrence formula

$$(2n+1)\,P_n(\mu) = \dot{P}_{n+1}(\mu) - \dot{P}_{n-1}(\mu)\,, \qquad n>0 \tag{28}$$

and (27), results in

$$(2n+1)(1-\mu^2)^{\frac{1}{2}}\,P_n(\mu) = P_{n+1}^1(\mu) - P_{n-1}^1(\mu)\,, \qquad n>0\,. \tag{29}$$

Introducing (29) into (26) yields

$$\int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}} P_n(\mu) I_1[t(\zeta_0^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}] e^{it\zeta_0\mu} d\mu$$
$$= \frac{2(i)^n}{2n+1} [P^1_{n+1}(\zeta_0) j_{n+1}(t) + H(n) P^1_{n-1}(\zeta_0) j_{n-1}(t)], \quad (30)$$

where $H(n)$ denotes the Heaviside delta function,

$$H(n) = \begin{cases} 0, & n \leq 0 \\ 1, & n > 0 \end{cases} \quad (31)$$

After substituting (27) and (30), in (25) the latter may be integrated to give the following expression:

$$T_2(x, \zeta_0, a) = -\frac{4}{\pi} \zeta_0(\zeta_0^2-1) \sum_{n=0}^{\infty} \frac{\overline{i^n}}{2n+1} A_n \int_0^{\infty} t \frac{K_1(ta)}{I_1(ta)} \times$$
$$\times [\dot{P}_{n+1}(\zeta_0) j_{n+1}(t) + H(n) \dot{P}_{n-1}(\zeta_0) j_{n-1}(t)] dt, \quad (32)$$

where $\overline{i^n}$ denotes the complex conjugate of i^n.

As in (18) let us define

$$T_2^m(\zeta_0, a) = \int_{-1}^{1} T_2(x, \zeta_0, a) P_m(x) dx. \quad (33)$$

Hence, using the relation [13]

$$\int_{-1}^{1} P_m(x) e^{itx} dx = 2i^m j_m(t), \quad (34)$$

there results from (32), (33) and (34), that

$$T_2^m(\zeta_0, a) = -\frac{8}{\pi} \zeta_0(\zeta_0^2-1) \sum_{n=0}^{\infty} \frac{\varepsilon_n^m}{2n+1} A_n \int_0^{\infty} t j_m(t) \frac{K_1(ta)}{I_1(ta)}$$
$$[\dot{P}_{n+1}(\zeta_0) j_{n+1}(t) + H(n) \dot{P}_{n-1}(\zeta_0) j_{n-1}(t)] dt, \quad (35)$$

where

$$\varepsilon_n^m = \begin{cases} -1, & m+n = 2k \\ 0, & m+n = k \qquad k = 1, 3, 5, \ldots, \\ 1, & m+n = 4k \end{cases} \quad (36)$$

and $\varepsilon_0^0 = 1$.

5. Solution for the vortex sheet strength

Application of the orthogonality properties of the Legendre polynomials to the integral equation (7) together with the definitions (18) and (33), results in the following relation:

$$4U\delta(m) = T_1^m(\zeta_0) + T_2^m(\zeta_0, a). \quad (37)$$

where $\delta(m)$ is zero if $m \neq 0$ and one if $m = 0$.

Substitution of (24) and (35) into (37) yields

$$-\frac{\dot{Q}_{2m+1}(\zeta_0)}{4m+1} A_{2m} = \frac{U\delta(2m)}{\zeta_0(\zeta_0^2-1)} + \sum_{k=1}^{\infty} Q_{2m+2k}(\zeta_0) A_{m+2k} + \sum_{n=0}^{\infty} D_{2n}^{2m}(\zeta_0, a) A_{2n}, \quad (38)$$

where

$$D_{2n}^{2m}(\zeta_0, a) = \frac{2(-1)^{m+n}}{(4n+1)\pi} \int_0^{\infty} t j_{2m}(t) \frac{K_1(ta)}{I_1(ta)} [\dot{P}_{2n+1}(\zeta_0) j_{2n+1}(t) +$$
$$+ H(n) \dot{P}_{2n-1}(\zeta_0) j_{2n-1}(t)] dt. \quad (39)$$

Only even integers should be considered for m in (37) since by symmetry $A_{2m+1} \equiv 0$ in (9). Equation (36) then implies that also n must be an even integer.

An alternative form of (38) is

$$\sum_{n=0}^{\infty} C_{2n}^{2m}(\zeta_0, a) A_{2n} = -\frac{U\delta(2m)}{\zeta_0(\zeta_0^2-1)}, \tag{40}$$

where

$$C_{2n}^{2m}(\zeta_0, a) = D_{2n}^{2m}(\zeta_0, a) + \delta(2n-2m)\frac{\dot{Q}_{2m+1}(\zeta_0)}{4m+1} + H(2n-2m)\,Q_{2n}(\zeta_0). \tag{41}$$

For the case of "zero blockage" (no walls) the right-hand side of (39) is identical zero (since $a \to \infty$) and the infinite set of linear equations (38) yields a rather simple solution

$$A_m = -\frac{U\delta(m)}{\zeta_0(\zeta_0^2-1)\dot{Q}_1(\zeta_0)}. \tag{42}$$

Introducing (42) into (3) and (9), yields the well-known solution [11] for the velocity distribution on the surface of a prolate spheroid in an infinite stream

$$\frac{u(x)}{U} = -\frac{1}{(\zeta_0^2-1)\dot{Q}_1(\zeta_0)}\left(\frac{\zeta_0^2-x^2}{\zeta_0^4-x^2}\right)^{\frac{1}{2}}. \tag{43}$$

A second case for which the system (40) has a trivial solution is for a slender spheroid whose cross sectional area is very small. In the limit where this area is zero, the spheroid shrinks unto a section of the x-axis between the two foci. This is equivalent to the limit of ζ_0 approaching the value of unity. For the particular case discussed above the solution of (40) is $A_m = U\delta(m)$ as expected. That is to say that in the limit the spheroid does not disturb the flow.

In most cases one is dealing with the case of "small blockage" (large values of a). For this case it is possible to replace the integral (39) by a series of tabulated functions. This can be done in the following manner.

The product of two spherical Bessel function may be expressed as an infinite power series in the argument [14].

$$j_m(t)j_n(t) = \sum_{k=0}^{\infty} B_{m,n}^k t^{m+n+2k}, \tag{44}$$

where

$$B_{m,n}^k = \frac{4(-1)^k}{k!}2^{m+n+2k}\frac{(m+n+2k+1)!\,(m+k+1)!\,(n+k+1)!}{(m+n+k+1)!\,(2m+2k+2)!\,(2n+2k+2)!}. \tag{45}$$

Using the following relation between the modified Bessel function of the first and second kind [13]

$$K_{m+1}(t)I_m(t) + K_m(t)I_{m+1}(t) = t^{-1}. \tag{46}$$

It can be shown by integration by parts that

$$(2k+1)a^{2k+1}\int_0^{\infty} t^{2k}\frac{K_1(ta)}{I_1(ta)}\,dt = \int_0^{\infty}\frac{t^{2k}}{I_1^2(t)}\,dt = \mathscr{I}(2k). \tag{47}$$

The function $\mathscr{I}$ has been computed numerically by Smythe [18] with an accuracy of eight decimal figures using Weddles integration rule. These functions are tabulated in [18] for forty-two sequential values of k.

The integral (39) expressed in terms of the $\mathscr{I}$ function reads

$$D_{2n}^{2m}(\zeta_0, a) = \frac{2(-1)^{m+n}}{(4n+1)\pi a}\sum_{k=0}^{\infty}\frac{1}{(2k+1)a^{2k}}\left[B_{2m,2n+1}^k \dot{P}_{2n+1}(\zeta_0)\mathscr{I}(2m+2n+2k+2) + H(n)B_{2m,2n-1}^k \dot{P}_{2n-1}(\zeta_0)\mathscr{I}(2m+2n+2k)\right]. \tag{48}$$

By examining the series (48) for large values of k it can be shown, based on the ratio convergence test ratio, that the above series converges uniformly for $a \geqq 1$, since [18]

$$\lim_{k\to\infty} \frac{\mathscr{I}(2k+2)}{\mathscr{I}(2k)} \to k^2 . \tag{49}$$

The convergence of (48) is quite good for $a > 1$; only a few terms are required to obtain the result with accuracy of several significant figures. For values of a such that $a < 1$ the coefficients D_{2n}^{2m} should be determined from (39) rather than (48). The advantage of using (48) on (39) is in the saving of the numerical integration and in the possibility of controlling the error by a proper truncation of the infinite series.

6. The longitudinal added mass coefficient

The longitudinal added mass coefficient of a body of revolution placed axisymmetric in a tube, is a parameter of primary importance in the analysis of the aerodynamic stability of bodies moving in a tube. Goodman [4] has calculated the added mass coefficient of a prolate spheroid using slender body theory. From the solution of (40) it is also possible to derive an exact expression for the longitudinal added mass coefficient of the spheroid.

The Taylor [19] added mass theorem yields an expression for the added mass coefficient of a Rankine body in terms of its image system which is assumed to be, in general, a combination of sources, sinks and doublets. This powerful theorem is not applicable for the present case since the body is generated by a distribution of vortex rings. However, following Lamb [11], a vortex ring is equivalent to a uniform distribution of doublets over the surface bounded by it. The axes of the doublets is taken to be normal to the surface everywhere and the density of the distribution is equal to the strength of the vortex divided by 4π. Hence, employing the Taylor added mass formula [19] for normal doublets, the following expression for the longitudinal added mass coefficient of the spheroid, here denoted by λ, is obtained.

$$UV(1+\lambda) = \pi\zeta_0(\zeta_0^2-1)\int_{-1}^{1} \gamma(\mu)(1-\mu^2)d\mu , \tag{50}$$

where V denotes the volume of the spheroid. Introducing (9) into (50) and using (15), yields the following simple form for the added mass coefficient

$$1+\lambda = A_0 - \tfrac{1}{5}A_2 . \tag{51}$$

For the case of a spheroid in an infinite stream, combination of (42) and (51) yields the well known expression for the longitudinal added mass coefficient of a prolate spheroid in an unbounded stream parallel to its major axis:

$$1+\lambda = -[\zeta_0(\zeta_0^2-1)\dot{Q}_1(\zeta_0)]^{-1} . \tag{52}$$

7. Numerical examples and discussions

The infinite system of linear equations for the coefficients A_m (40), was solved by using the method of reduction [20]. In this method the solution is found by solving a sequence of finite systems, each of which is obtained from the infinite set by discarding all equations and unknowns beyond a certain number N. According to Kantorovich and Krylov [20] the solution of the finite system will converge, under certain conditions, to the solution of the infinite system. It was rather difficult to show that the conditions for the convergence theorem are satisfied and therefore it was assumed that if the finite system has a solution it must converge to the solution of the infinite system. A numerical verification of the above assumption is obtained when the number N of equations kept is increased. The coefficients A_m approach a limiting value.

The number N was chosen so as to permit maximum error of 10^{-6} between two successive approximations. Denote the numerical value of A_m obtained from the solution of N equations

TABLE 1

The convergence of the coefficients A_n versus N $(a=l=6b)$★

	$N=0$	$N=1$	$N=2$	$N=3$	$N=4$
A_0	1.05752143	1.05738838	1.05739913	1.05739913	1.05739915
$-A_2\times10^3$		4.44841151	4.39213028	4.39520394	4.39510117
$A_4\times10^4$			5.20406031	5.14876887	5.15061879
$-A_6\times10^5$				3.72772389	3.70100377
$A_8\times10^6$					1.54243522
$\sum_{n=0}^{N} A_n$	1.05752143	1.05961259	1.05979096	1.05980146	1.05980184

$A_0 = 1.04518289$ ($l=6b$, no duct)

★ $2l$ = Length of spheroid

$2b$ = Maximum diameter of spheroid

$2a$ = Diameter of duct.

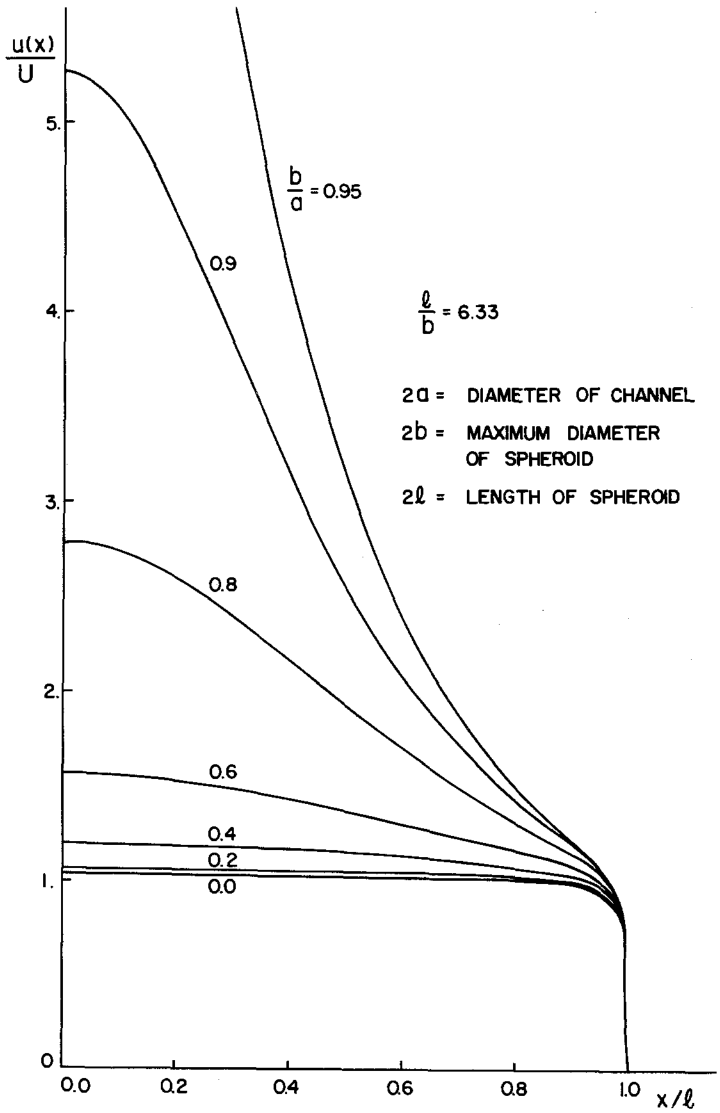

Figure 1. Velocity distribution on the surface of a spheroid for different blockages.

by $A_m^{(N)}$ and similarly, the value found by solving $N+1$ equations by $A_m^{(N+1)}$. The error criterion used here was to choose N such that the following two conditions

$$|A_m^{(N+1)} - A_m^{(N)}| < 10^{-6}, \tag{53}$$

$$\left|\sum_{n=0}^{N+1} A_n^{(N+1)} P_n(0) - \sum_{n=0}^{N} A_n^{(N)} P_n(0)\right| < 10^{-6}, \tag{54}$$

are satisfied.

Since $A_1 > 1$, the inequality (53) implies that the coefficients A_m are computed with accuracy of seven decimal figures. Similarly, (54) suggests that the velocity on the spheroid is computed with the same accuracy.

For small and moderate blockage the rate of convergence of $A_m^{(N)}$ to its limiting value was fast, i.e. only small values of N are needed to obtain this accuracy. The value of N is increasing with an increase of the blockage. A numerical example of the coefficients $A_m^{(N)}$, obtained by solving (40) for different m and N, is given in Table 1. The spheroid selected for this example, is

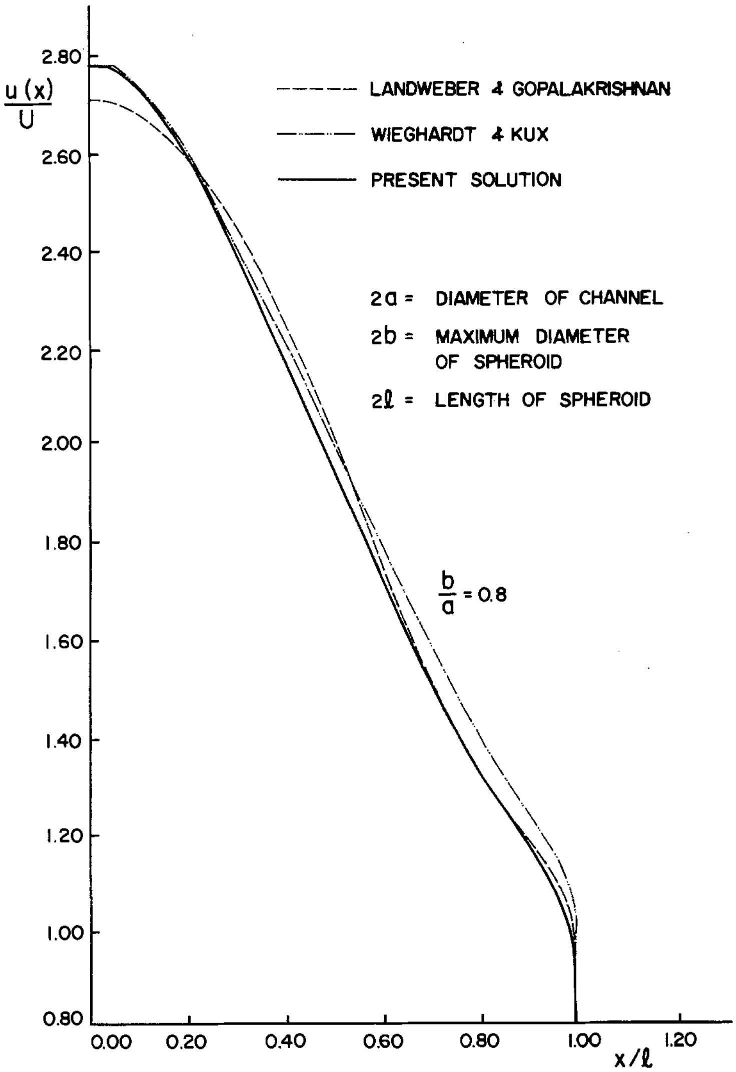

Figure 2. A comparison between the present solution for the velocity distribution and the solutions given in [5] and in [8].

TABLE 2

Final solution for the coefficients A_n versus b/a ($l/b = 6.33$)★

b/a	A_0	$-A_2$	A_4	$-A_6$	A_8	$-A_{10}$	A_{12}	$-A_{14}$	A_{16}	$-A_{18}$	A_{20}	$-A_{22}$	A_{24}	$-A_{26}$	A_{28}
0	1.045183														
0.2	1.061639	0.009332	0.001336	0.000101	0.000002										
0.4	1.152135	0.090622	0.0175037	0.000219	0.000342	0.000017	0.000021	0.000002							
0.6	1.368470	0.340641	0.082149	0.003326	0.000372	0.001017	0.000118	0.000293	0.000002						
0.8	1.959493	1.199103	0.463771	0.115657	0.0397407	0.013401	0.000130	0.001139	0.000908	0.001670	0.001184	0.000305			
0.9	2.882610	2.833136	1.586921	0.713980	0.336933	0.148698	0.055719	0.022437	0.013142	0.009657	0.005441	0.001966	0.000462		
0.95	4.253825	5.608938	4.077351	2.528613	1.541218	0.897951	0.502392	0.283147	0.165665	0.098803	0.056176	0.029258	0.0135792	0.005235	0.000996

★ $2l$ = Length of spheroid
$2b$ = Maximum diameter of spheroid
$2a$ = Diameter of duct.

that used by Satija [3], having length-diameter ratio of 6 and the same value for the ratio of the diameters of duct and spheroid.

The coefficients D_n^m in (39) are suited for computation by Laguerre quadrature formula. The Laguerre 15-point formula [13] was used.

The procedure for calculating the velocity distribution on the surface of a spheroid is illustrated for the spheroid used by Kux and Wieghardt [8] and by Landweber and Gopalakrishnan [5], with length-diameter ratio of 6.33. The system of equations (40) was solved for different blockages and the velocity distribution was computed from (3). The numerical results are plotted in Figure 1 and the values of the coefficients A_m for different blockages are given in Table 2. Figure 2 is a comparison between the results presented in [5] and in [8] and the present solution for a ratio of channel diameter to spheroid diameter equal to 1.25. From this comparison it appears that Kux and Wieghardt's solution yields a better accuracy than Landweber and Kopalakrishnan's solution in the central portion of the body. The opposite statement is true for the region near the stagnation points where Landweber and Gopalakrishnan method seems to be the more accurate one. Some numerical difficulties arise when these two methods are used for blockages which are larger than 0.8. This is not the case in the present method. The solution was found to be convergent even for blockage of 0.95. For very small blockage the correction to the velocity distribution on the surface due to the presence of the walls is insignificant. For the moderate blockage of 0.4 the maximum correction for the unbounded case velocity distribution is less than 15 percent. This correction is rapidly increasing with further

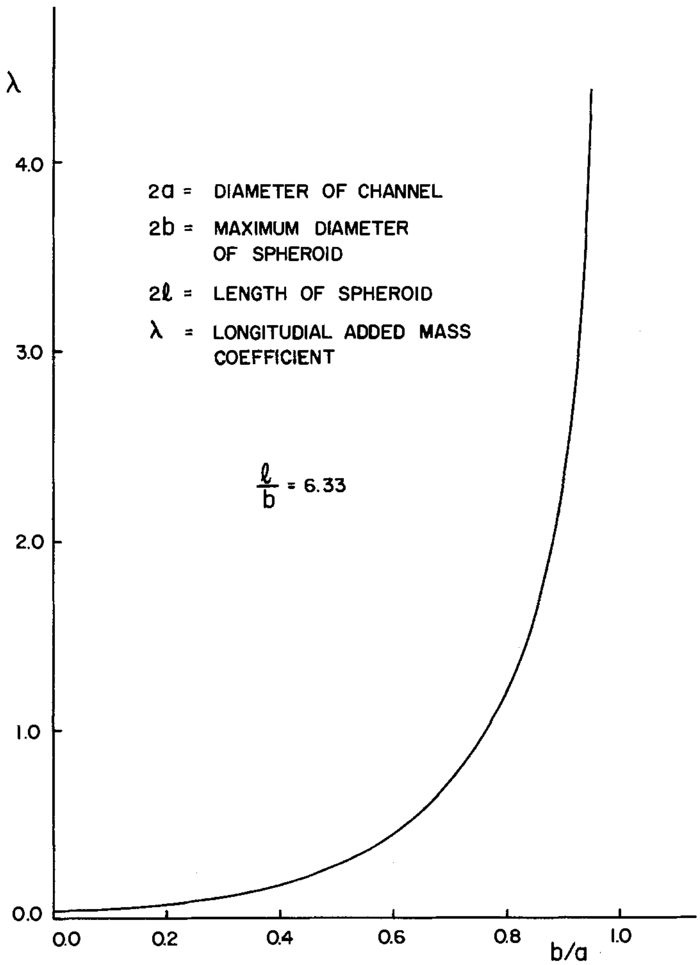

Figure 3. The longitudinal added mass coefficient of a spheroid *versus* the blockage of the channel.

increase of the blockage, reaching a value of 180 percent for blockage of 0.8 and 430 percent for blockage of 0.9.

The longitudinal added mass coefficient as computed from (51) for the same spheroid is plotted in Figure 3 for different blockages. This coefficient is increasing exponentially with increasing the value of the blockage approaching the value of infinity as the blockage is approaching unity. The numerical values of $1+\lambda$ are approximately equal to the value of $u(0)/U$ for the same blockage. This statement is exact in the case of zero blockage.

As a concluding remark it should be mentioned that the present solution may also be applied to bodies with shapes close to a spheroid by defining an "equivalent spheroid", i.e. a spheroid having the same volume and the same maximum cross sectional area as the original body.

Acknowledgment

The author is indebted to Professor L. Landweber for introducing him this problem. The author would like to dedicate this paper to Professor K. Wieghart on the occasion of his sixtieth birthday.

REFERENCES

[1] P. Levine, Incompressible Potential Flow About Axially Symmetric Bodies in Ducts, *Journal of Aeronautical Sciences*, 25 (1958).
[2] G. N. Watson, The Use of Series of Bessel Functions in Problems Connected with Cylindrical Wind-Tunnels, *Proc. Roy. Soc. A.*, 130 (1931).
[3] K. S. Satija, Potential Flow About Body in Finite Stream, *Journal of Hydronautics*, 7, 1 (1973).
[4] T. R. Goodman, Aerodynamic Characteristics of a Slender Body Travelling in a Tube, *AIAA Journal*, 9, 4 (1971).
[5] L. Landweber and K. Gopalakrishnan, *Irrotational Axisymmetric Flow About Body of Revolution in Circular tube*, *Schiffstechnik* 20, 102 (1973).
[6] J. Mathew, *Theoretical and Experimental Investigation of the Incompressible Flow Past Axi-Symmetric Bodies in Ducts*, Ph.D. Thesis, Department of Applied Mechanics, Indian Institute of Technology, Madaras (1970).
[7] A. Klein and J. Mathew, Incompressible Potential Flow Solution for Axisymmetric Body-Duct Configurations, *Z. Flugwiss.*, 20, Heft 6 (1972).
[8] J. Kux and K. Wieghardt, *Blockage Effect on the Frictional Drag of a Spheroid*, Thirteenth International Towing Tank Conference, Berlin–Hamburg (1972).
[9] J. L. Hess and A. M. O. Smith, Calculation of Potential Flow About Arbitrary Bodies, *Progress in Aeronautical Sciences*, 8, Pergamon Press, New York (1966).
[10] E. Hellinger and O. Toeplitz, Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten, *Encyklopädie der Mathematischen Wissenschaften* II, C 13 (1928) 1349.
[11] H. Lamb, *Hydrodynamics*, Dover Publishing Co. (1932).
[12] L. Landweber, *The Axially Symmetric Potential Flow About Elongated Bodies of Revolution*, TMB Report 761, August 1951.
[13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55 (1964).
[14] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 4th Edition (1935).
[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Publishing Co. (1953).
[16] T. H. Havelock, *Collected Papers on Hydrodynamics*, Office of Naval Research v.s. Government Printing Office, Washington D.C. (1963) 591.
[17] G. N. Watson, *Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press (1958).
[18] W. R. Smythe, Flow Around a Sphere in a Circular Tube, *Physics of Fluids*, 4 (1961).
[19] G. I. Taylor, The Energy of Body Moving in an Infinite Fluid With An Application to Airships, *Proc. Roy. Soc. A.*, 120 (1928).
[20] L. V. Kantorovich and V. L. Krylov, *Approximate Methods of Higher Analysis*, Interscience Publishers, New York and P. Noordhoff Ltd., Groningen, The Netherlands (1958).